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THE FORMULATION AND SOLUTION OF SOME PROBLEMS OF THE THEORY OF INVESTMENT WITH OCCASIONAL PRIVATE BENEFITS

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The article discusses an alternative investment model with given probabilities of investment success and with a random distribution function of the benefits from them. A number of problems are formulated that most often arise when using this mathematical model. Submitted and solved the problem of finding the threshold for a given contract. As a result, it is shown that it is necessary to establish the value of the investment slightly higher than the threshold from which the entrepreneur evades the contract. It is shown that in exceptional cases, if an entrepreneur is indifferent between evasion and work, he decides to work. The task of finding the maximum amount of debt for a given investment threshold is investigated. It is assumed that the investor knows that if the volume is less than the investment threshold, then the entrepreneur works, and if the amount of debt is greater than the investment volume, then the entrepreneur evades, because the expected return on investment must be positive, and the investor must be insured to agree to the contract. The expected utility of an entrepreneur for a given investment threshold has been determined. The conditions were found under which the contract is optimal for an entrepreneur (subject to the break-even of investors). It is shown that in order to stimulate an entrepreneur to work, the "threshold remuneration" should be quite high (somewhere in the upper half of the expected profitability of the entrepreneur). The situation in which the private benefit is observable and verifiable is investigated. Solved the problem of determining the optimal contract between the entrepreneur and investors (compensation can be made depending on the level of private benefits). The proposed conditions for the adjustment of the contract with increasing benefits. It is shown that the optimal contract for an entrepreneur implies zero return for the investor. An analytical solution of these problems is given taking into account the stated requirements and limitations.

Keywords: alternative investment model, entrepreneur's expected utility, investment.

About problem

In the work on corporate finance, various mathematical models are proposed and analyzed [1-6]. In this paper we consider the variable-investment model proposed in [7].

Description of the mathematical model

An entrepreneur initially has cash *A*. For investment I, the project yields RI in the case of success and 0 in the case of failure. The probability of success is equal to $p_H \in (0, 1)$ if the entrepreneur works and $p_L = 0$ if the entrepreneur shirks. The entrepreneur obtains private benefit BI when shirking and 0 when working. The per-unit private benefit B is unknown to all ex ante and is drawn from (common knowledge) uniform distribution F:

$$P_r(B < \dot{B}) = F(\dot{B}) = \dot{B}/R$$

for $\dot{B} \leq R$, with density $f(\dot{B}) = 1/R$. The entrepreneur borrows $I - A$ and pays back $R_1 = r_1 I$ in the case of success. The timing is the following: the contract (I, r_1) is signed; the entrepreneur learns about B privately and afterwards he chooses effort. In the end, income is realized and payments are made.

Task 1. For a given contract (I, r_1), find the threshold B^*, i.e., the value of the private per-unit benefit above which the entrepreneur shirks.

Solution. Entrepreneur's expected utility in case he works for a given B:

$$E_w[U] = p_H RI + (1 - p_H)\cdot 0 - p_H r_1 I - (1 - p_H)\cdot 0 = p_H I(R - r_1).$$

Entrepreneur's expected utility in case he shirks for a given B:

$$E_s[U] = 0 + BI - 0 = BI.$$

At this point:

$E_s[U]$ – expected utility of the entrepreneur if she shirks (i.e. doesn't work),

$E_w[U]$ – expected utility of the entrepreneur if she works.

So, after seeing B, the entrepreneur will decide to shirk if:

$$E_s[U] \geq E_w[U]$$

$$\rightarrow BI \geq p_H I(R - r_1)$$

$$\rightarrow B^* = p_H(R - r_1).$$

Note: hereafter we will assume that if $B = B^*$, so if the entrepreneur is indifferent between shirking and working, he decides to work.

Task 2. For a given B^* (or equivalently r_1, which determines B^*), find the debt capacity. Find value of B^* (or r_1) that is debt capacity highest.

Solution. The investor knows that if $B \leq B^*$, then the entrepreneur works, and if $B > B^*$, then the entrepreneur shirks, so, his expected profit from the investment is as follows, and it should be nonnegative for the investor to agree to a contract:

$$E[Pr_{inv}] = Pr(B \leq B^*)E_w[Pr_{inv}] + Pr(B > B^*)E_s[Pr_{inv}] - (I - A)=$$

$$=(B^*/R)p_H r_1 I + (1 - B^*/R)\cdot 0 - (I - A) = (B^*/R)p_H r_1 I - I + A \geq 0.$$

At this point $E[Pr_{inv}]$ – expected utility of the investor.

Solving this inequality we get the following maximum debt capacity:

$$(I - A)_{max} = Ap_H B^* r_1/(R - p_H B^* r_1) = Ap^2_H(R - r_1)r_1/(R - p^2_H(R - r_1)\, r_1).$$

(!) Note: we assumed that $R - B^* p_H r_1 = R - p^2_H(R - r_1)r_1 > 0$ as otherwise the debt capacity would be infinitely large.

Now we maximise $(I - A)_{max}$ with respect to r_1 to get the value for which the debt capacity is the highest:

$$\partial(I\text{-}A)_{max} / \partial r_1 = (Ap^2_H(R - 2r_1)(R - p^2_H(R - r_1)\, r_1) +$$

$$+ p^2_H(R - 2r_1)(Ap^2_H(R - r_1)r_1)/(R - p^2_H(R - r_1)r_1)^2 =$$

$$= Ap^2_H R(R - 2r_1)/(R - p^2_H(R - r_1)r_1)^2 = 0$$

$$\rightarrow r_1^* = R/2.$$

Note that the numerator of $\partial(I - A)_{max}/\partial r_1$ is decreasing in r_1 and the denominator is increasing, hence, the whole function is decreasing in r_1, which means that we, indeed, found maximum.

Task 3. Determine the entrepreneur's expected utility for a given B^*. Find the conditions under which the contract is optimal for the entrepreneur (subject to the investors breaking even).

Solution. We find entrepreneur's expected utility analogously to that of the investor:

$$E[U] = P(B \leq B^*)Ew[U] + P(B > B^*)Es[U] =$$

$$= (B^*/R)p_H(R - r_1)I + (1 - B^*/R)E[B| > B^*]I =$$

$$= (B^*/R)p_H(R - r_1)I + ((R - B^*)/R)\cdot((R - B^*)/2)I =$$

(*substituting* $B^* = p_H(R - r_1)$)

$$= (B^*)^2 I/R + (B\text{-}B^*)^2 I/(2R) = I/(2R)(3(B^*)^2 - 2B^*R + R^2).$$

And the constraint is the investors breaking even which is his expected value is nonnegative (calculated in Task 2):

$$E[Pr_{inv}] = (B^*/R)p_H r_1 - 1)I + A = (B^*(p_H R - B^*)/R - 1)I + A \geq 0$$

$$\rightarrow I \leq AR/(R - B^*(p_H R - B^*)).$$

Now we need to show that

$$0.5p_H R < B^* < p_H R.$$

First, the right-hand side inequality. It simply follows from the fact that $r_1 > 0$:

$$B^* = p_H(R - r_1) < p_H R.$$

Second, the left-hand side inequality. We can see that $B^* = 0.5p_H R \leftrightarrow r_1^* = R/2$, which is the value when the debt capacity is the highest. But giving the largest possible debt is not optimal, it is analogous to maximizing revenue instead of profit, so the optimal value lies above this one.

The inequality shows that in order to incentivize the entrepreneur to work the "threshold benefit" should be set quite high (to somewhere in the upper half of the expected yield of the entrepreneur).

Task 4. Suppose now that the private benefit B is observable and verifiable. Determine the optimal contract between the entrepreneur and the investors (the reimbursement can now be made contingent on the level of private benefits):

$R_1 = r_1(B)I)$.

Solution. Now, we can treat B as just a variable and we need to find a contract $(I(B); r_1(B))$ that is optimal for the entrepreneur.

If she works, her expected utility is:

$$E_w[U] = p_H(R - r_1(B))I(B).$$

If she shirks, her expected utility is:

$$E_s[U] = BI(B).$$

Hence, for her to work the following condition should be satisfied:

$$p_H(R - r_1(B)) \geq B$$

$$\Rightarrow r_1(B) \leq R - B/p_H. \quad (*)$$

Then, the payoff of the investor if the entrepreneur works is:

$$E_w[Pr_{inv}] = r_1(B)I(B) - I(B) + A.$$

And her payoff if the entrepreneur shirks is:

$$E_w[Pr_{inv}] = -I(B) + A < 0.$$

Hence, the investor would never agree to a contract where the entrepreneur shirks, so (*) has to be satisfied.

Also, since $r_1(B)I(B)$ goes from entrepreneur to investor, then if the investor gets strictly positive payoff, they can rewrite contract decreasing $r_1(B)$ thus increasing the utility of the entrepreneur till investor's payoff is non-negative. So, the optimal contract for the entrepreneur implies zero payoff for the investor, i.e.:

$$r_1(B)I(B) - I(B) + A = 0$$

$$\Rightarrow I(B) = A/(1 - r_1(B)). \quad (**)$$

Now we maximize expected utility of the entrepreneur if she works taking into account (**) and the constraint (*):

$$\max_{r1(B)}[Ew[U]] = \max_{r1(B)}[\, p_H(R - r_1(B))I(B)] =$$

$$= \max_{r1(B)}[\, p_H A(R - r_1(B))/(\, 1 - r_1(B))] =$$

$$= \max_{r1(B)}[\, p_H A(1 + (R - 1)/(\, 1 - r_1(B)))]\ .$$

Since R is gross return on investment, we can assume that R > 1, therefore, $E_w[U]$ is a monotonously increasing function of $r_1(B)$, so we choose the largest $r_1(B)$ that our constraint (*) allows, hence

$$r_1^*(B) = R - B/p_H.$$

And the optimal investment size:

$$I^*(B) = A/(1 - r_1^*(B)) = p_H A/(\, p_H(1 - R)+B) .$$

So, the optimal contract is as follows:

$$(I^*(B),\, r_1^*(B)) = (\, p_H A/(\, p_H(1 - R)+B),\ R - B/p_H\,).$$

Task 5. Investors have utility over consumption relative to a reference point X_t:

$$u_t = ((C_t - X_t)^{1-\gamma} - 1)/(1 - \gamma).$$

Consumption growth is i.i.d Normal:

$$g_{t+1} = g+v_{t+1},\ \ Var(v_{t+1}) = \sigma_v^2.$$

Define surplus consumption S_t :

$$S_t = (C_t - X_t)/\, C_t.$$

Log surplus consumption is driven by consumption news:

$$s_{t+1} = \hat{s}+\phi(s - \hat{s}) + \lambda(s_t)v_{t+1},$$

where the sensitivity function is specified:

$$\lambda(s_t)=(1 - 2(s- \hat{s}))^{0.5}/\, \hat{s} - 1,$$

when $s_t < s_{max}$ and 0 otherwise.
The specification of $\hat{s} = \log\hat{S}$:

$$\hat{S}= \sigma_v\, (\gamma/(1 - \phi - b/\gamma)^{0.5},$$

where b is the preference parameter.

Show that the one-period real risk-free rates are now time-varying, and are linear in the consumption surplus s_t.

Consider the case of b > 0. How do the interest rates vary with the consumption surplus ratio? Are they low or high in "good times"?

Compare the implications relative to part *b*.

In long-run risks model, do real rates fall or rise in "good" times.

Can you use this model predictions to test the two asset-pricing theories?

Solution.

First, we calculate the SDF:

$$M_{t+1}=\beta u_{t+1}(C_{t+1})/\, u_t(C_t)= \beta((C_{t+1} - X_{t+1})/\, (C_t - X_t))^{-\gamma}=$$

$$= \beta(S_{t+1} / (S_t\,)^{-\gamma}(C_{t+1} / (C_t\,)^{-\gamma}=$$

$$=\beta e^{-\gamma(st+1\text{ - }st)}e^{-\gamma gt+1}= \beta e^{-\gamma((st\text{ - }\hat{s})(\phi\text{-}1)+\lambda(st)\ vt+1)}\ e^{-\gamma gt+1}=$$
$$=\beta e^{-\gamma((st\text{ - }\hat{s})(\phi\text{-}1)}e^{-\gamma gt}e^{-\gamma\ (\lambda(st)+1)vt+1)}.$$

Note that since $v_{t+1} \sim N(0, \sigma_v^2)$, then

$$-\gamma\ (\lambda(s_t)+1)v_{t+1}) \sim N(0, \gamma^2(\lambda(s_t)+1)^2\sigma_v^2).$$

Now we can find the risk-free rate via the Euler equation:

$$r_{t+1}=\log(1/E_t[M_{t+1}])= -\log\beta+\gamma((s_t-\hat{s})(\phi-1)+\gamma g_t-\log E_t[e^{-\gamma\ (\lambda(st)+1)vt+1)}]=$$
$$= -\log\beta+\gamma((s_t-\hat{s})(\phi-1)+\gamma g_t-0.5\gamma^2(\lambda(s_t)+1)^2\sigma_v^2)=$$
$$= -\log\beta+\gamma((s_t-\hat{s})(\phi-1)+\gamma g_t-0.5\gamma^2\sigma_v^2(1-2(s_t-\hat{s}))(1-\phi-b/\gamma)/(\sigma_v^2\gamma)=$$
$$= -\log\beta+\gamma g_t+\gamma(\phi-1)/2+b/2-b(s_t-\hat{s}).$$

Indeed, we see that the risk-free rate now depends on s_t making it time-varying, and the relationship is linear.

If $b > 0$, then the formula above shows that the risk-free rate and s_t are adversely related. Hence, in "good times", i.e. when the consumption surplus ratio is high, interest rates are low.

Let $b < 0$. Here, interest rates are increasing in s_t and are therefore pro-cyclical. This means that they are high when consumption surplus is high, and low when consumption surplus is low. In this, the precautionary savings effect dominates, as interest rates rise in good times to prevent excess consumption and fall in poor times to prevent excess savings.

Results

We see that the models give opposite relationships between the risk-free rate and the overall economic situation. Hence, to test both models we can look at this relationship in the data and then pick the model which has the same relationship as in the data. One way to find the relationship in the data is to define the "bad times" as the periods of stagnation and the "good times" as all other periods (or, perhaps, all other periods except for the ones which are the closest to the periods of stagnation). Then we just compare the risk-free rates in both good and bad times and see which ones are larger.

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ПОСТАНОВКА И РЕШЕНИЕ НЕКОТОРЫХ ЗАДАЧ ТЕОРИИ ИНВЕСТИЦИЙ ПРИ СЛУЧАЙНЫХ ЧАСТНЫХ ВЫГОДАХ

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В статье рассматривается альтернативная инвестиционная модель с заданными вероятностями успеха инвестиций и со случайной функцией распределения выгоды от них. Формулируется ряд задач, наиболее часто возникающих при использовании данной математической модели. Предложена задача поиска порога для заданного контракта. Как следствие, показано, что необходимо установить стоимость инвестиции чуть выше, чем порог, начиная с которого предприниматель уклоняется от контракта. Показано, что в исключительных случаях если предприниматель безразличен между уклонением и работой, он решает работать. Исследована задача поиска максимального объема задолженности для заданного порога инвестиций. Предполагается, что инвестор знает, что если объем меньше порога инвестиций, то предприниматель работает, а если объем задолженностей больше объема инвестиций, то предприниматель уклоняется, поскольку ожидаемая прибыль от инвестиций должна быть положительной, и инвестор должен быть подстрахован, чтобы согласиться на контракт. Определена ожидаемая полезность предпринимателя для данного порога инвестиций. Найдены условия, при которых договор является оптимальным для предпринимателя (при условии безубыточности инвесторов). Показано, что для стимулирования предпринимателя к работе «пороговое вознаграждение» должно быть достаточно высоким (где-то в верхней половине ожидаемой доходности предпринимателя). Исследована ситуация, в которой частная выгода поддается наблюдению и проверке. Решена задача определения оптимального договора между предпринимателем и инвесторами (возмещение может быть сделано в зависимости от уровня частных выгод). Предложены условия корректировки контракта при увеличении выгоды. Показано, что, оптимальный контракт для предпринимателя подразумевает нулевую отдачу для инвестора.

Приводится аналитическое решение этих задач с учетом заявленных требований и ограничений.

Ключевые слова: альтернативная инвестиционная модель, ожидаемая полезность предпринимателя, инвестиции.

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